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Kähler Anomalies in Supergravity and Flux Vacua

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Abstract

We review the subject of Kähler anomalies in gauged supergravity, emphasizing that field equations are inconsistent when the Kähler potential is non-invariant under gauge transformations or when there are elementary Fayet-Iliopoulos couplings. Flux vacua solutions of string theory with gauged $U(1)$ shift symmetries appear to avoid this problem. The covariant Kähler anomalies involve tensors which are composite functions of the scalars as well as the gauge field strength and space-time curvature tensors. Anomaly cancellation conditions will be discussed in a sequel to this paper.

1 Introduction

This paper is devoted to the subject of Kähler anomalies in gauged $\mathcal{N} = 1$, $D = 4$ supergravity theories. The subject is certainly not new, but we revisit it because gauged supergravity appears as the effective four-dimensional theory in flux compactifications of string theory and in recent applications of Fayet-Iliopoulos couplings.[1] Kähler anomalies are not always physically significant, but when they are the field equations of the theory become inconsistent. The relation between anomalies and inconsistency is emphasized in our work.

The models we consider contain the supergravity multiplet (e^i_μ, Ψ_μ), coupled to gauge multiplets (A^a_μ, λ^a) and chiral multiplets (z^α, ψ^α). The dynamics of the chiral multiplets is that of a non-linear σ-model whose target space is an n-dimensional Kähler manifold called $\mathcal{T}$. In these theories the (Majorana) gravitino covariant derivative contains both the spin connection and a $U(1)$ axial gauge connection, i.e.

$$D_\mu \Psi_\nu = \Big(\partial_\mu + \frac{1}{4}\omega_{\mu ij}\gamma^{ij} + \frac{1}{2} i B_\mu \gamma_5\Big)\Psi_\nu \ . \tag{1.1}$$

The Kähler connection B_μ is typically a composite function of the scalars z^α, $z^{\bar\alpha}$ and the elementary vectors A^a_μ. The structure of supergravity requires the Kähler connection to couple to the gravitino and all other fermion fields.

The Kähler connection and the conventional axial couplings of the fundamental gauge field A^a_μ lead to anomalies of gauge currents which we explore through their effects on the consistency condition for gauge field equations of motion[2]

$$D_\mu \overset{a}{F}{}^{\mu\nu} = \overset{a}{J}{}^{\nu}. \tag{1.2}$$

The left side vanishes identically if one applies a further D_ν,

$$0 \equiv D_\nu D_\mu \overset{a}{F}{}^{\mu\nu} = D_\nu \overset{a}{J}{}^{\nu}, \tag{1.3}$$

and the current must be conserved for consistency. If the classical action is gauge invariant, then the current is conserved classically for field configurations which satisfy classical equations of motion. But, as is very well known, fermion triangle anomalies, which yield (schematically)

$$D_\nu \overset{a}{J}{}^{\nu} \ \propto \ \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}[\overset{a}{T} F_{\mu\nu} F_{\rho\sigma}]\,, \tag{1.4}$$

[1] For early supergravity models in which the relevant structure appears see [1, 2, 3].

[2] Gauge group indices are denoted by superscripts in the text and by "overscripts" in equations.

spoil current conservation and consistency at the quantum level [4].

It is the consistency condition (1.3) that we study in supergravity models. By classical manipulation we express $D_\nu J^{a\nu}$ in terms of bilinear fermion currents, and we evaluate their anomalies, using the Fujikawa method [5] to express results in terms of covariant anomalies. The analysis is a quite complicated affair in the supergravity models, so we develop the basic ideas in simpler truncated models in section 3 before applying them to the general situation in section 4. This follows a review of gauged Kähler non-linear σ-models in section 2.

In section 5, we turn our attention to supergravity models which descend from flux compactifications of superstring theory. We study a generic model with gauge shift symmetry and show that the field equations are consistent although there are uncanceled triangle anomalies. The question of anomaly cancellation in other models then remains. For gauge current anomalies this requires the conversion of covariant to consistent anomalies. This is done in a class of models which include Fayet-Iliopoulos couplings in a sequel to this paper [6].

2 Kähler manifolds and holomorphic isometries

We begin by reviewing (and defining notation for) the local geometry of Kähler manifolds and their continuous isometries. For more information, see [7, 8, 9]. We also discuss Kähler anomalies in general terms.

The scalar fields are complex coordinates on $\mathcal{T}$. They are denoted collectively by z^A, $A = 1, \dots, 2n$, which split into n holomorphic coordinates z^α and n anti-holomorphic $z^{\bar\alpha}$. The metric splits in the standard fashion

$$ds^2 = G_{AB}dz^A dz^B = 2G_{\alpha\bar\beta}dz^\alpha dz^{\bar\beta} \ , \tag{2.1}$$

in which the metric tensor can be expressed as second derivative of the Kähler potential $K(z,\bar z)$, viz.

$$G_{\alpha\bar\beta} = K,_{\alpha\bar\beta} = \frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} K(z,\bar z) \ . \tag{2.2}$$

We use a comma to denote partial derivatives, and a semi-colon for Kähler covariant derivatives, e.g $V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma}V^\gamma$. The only non-vanishing components of the Christoffel connection Γ^A_{BC} are the all-holomorphic $\Gamma^\alpha_{\beta\gamma} = G^{\alpha\bar\delta}G_{\gamma\bar\delta},_\beta$ and its complex conjugate $\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}$. The curvature tensor $R_{AB}{}^C{}_D$ enjoys the usual symmetries, but the only non-vanishing components are $R_{\alpha\bar\beta}{}^\gamma{}_\delta = -\Gamma^\gamma_{\alpha\delta},_{\bar\beta}$ and those related by symmetry and complex conjugation.

It is significant that the metric $G_{\alpha\bar{\beta}}$ does not change under Kähler transformations of the potential

$$K(z,\bar{z}) \;\rightarrow\; K'(z,\bar{z}) = K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}) \ . \tag{2.3}$$

In general the potential is not a global scalar, but is locally defined in each coordinate chart, the definitions in overlapping charts related by a Kähler transformation. Most terms in gauged supergravity Lagrangians are invariant under (2.3), but the Kähler connection changes by $B_\mu \rightarrow B_\mu + \partial_\mu \mathrm{Im}(f(z))$. Classically this can be compensated by an axial gauge transformation of the fermions, e.g. $\Psi_\mu \rightarrow \exp(-\frac{i}{2}\mathrm{Im}(f(z))\gamma_5)\Psi_\mu$, but this transformation is anomalous. This is the basic Kähler anomaly. It does not necessarily make the theory inconsistent at the quantum level. In this paper we focus on situations in which a gauge transformation of the bosonic fields induces the Kähler transformation (2.3). The accompanying fermion gauge transformation is anomalous and the theory does become inconsistent. We now review the machinery needed to implement gauge symmetry.

In our physical models gauge fields couple to holomorphic isometries of $\mathcal{T}$. Each such isometry is defined by a holomorphic Killing vector $X^{a\alpha}(z)$. There is a real scalar Killing potential $D^a(z,\bar{z})$ related to each X^a_α by $D^a_{,\alpha} = -iX^a_\alpha$. The holomorphic Killing vectors satisfy $X^a_{\alpha;\bar{\beta}} + X^a_{\bar{\beta};\alpha} = 0$ and generate a Lie algebra via the Lie bracket relations

$$\overset{a}{X}{}^{\beta}\,\overset{b}{X}{}^{\alpha}{}_{,\beta} - \overset{b}{X}{}^{\beta}\,\overset{a}{X}{}^{\alpha}{}_{,\beta} = f^{abc}\,\overset{c}{X}{}^{\alpha} \ . \tag{2.4}$$

We assume that this Lie algebra is a direct sum of a compact simple algebra and possible $U(1)$ subalgebras. The Killing potentials in the non-abelian simple sector are uniquely defined by the requirement that they transform in the adjoint representation, i.e.

$$\overset{a}{X}{}^{\alpha}\overset{b}{D}{}_{,\alpha} + \overset{a}{X}{}^{\bar{\alpha}}\overset{b}{D}{}_{,\bar{\alpha}} = f^{abc}\,\overset{c}{D} \ , \tag{2.5}$$

while those in abelian factors are defined up to an additive integration constant, i.e., $D^a \rightarrow D^a + \xi^a$, where ξ^a is a Fayet-Iliopoulos coupling in the physical context.

The Killing vector relations above state the fact that the Kähler metric $G_{\alpha\bar{\beta}}$ is invariant under the isometry, and the connection and curvature tensor are also invariant. However, it is important to note that the Kähler potential need not be invariant. It must satisfy only the weaker condition

$$\overset{a}{X}{}^{\alpha}K_{,\alpha} + \overset{a}{X}{}^{\bar{\alpha}}K_{,\bar{\alpha}} = \overset{a}{F}(z) + \overset{a}{\bar{F}}(\bar{z}) \tag{2.6}$$

which implies that the metric is invariant. The holomorphic quantity $F^a(z)$ will play an important role in our discussion. It is easy to show that D^a and F^a are related by

$$\overset{a}{D} = i(K_{,\alpha}\overset{a}{X}{}^{\alpha} - \overset{a}{F}) = -i(K_{,\bar{\alpha}}\overset{a}{X}{}^{\bar{\alpha}} - \overset{a}{\bar{F}}) \ . \tag{2.7}$$

Note that even in the case of an invariant Kähler potential (2.6) always leaves the freedom to add an imaginary constant to F^a. In the non-abelian case, the value of that constant is fixed by imposing (2.5) while for an abelian isometry it reflects the freedom to add a Fayet-Iliopoulos coupling ξ^a to the theory. Even when the Kähler potential is gauge invariant and the right side of (2.6) vanishes, there are Kähler anomalies which threaten the consistency of the supergravity theory if $\xi^a = \mathrm{Im}(F^a) \neq 0$.

In gauged supergravity all fermion fields couple to the composite Kähler connection which is given by

$$B_\mu = \frac{1}{2i}\Big(K_{,\alpha}\, D_\mu z^\alpha + \overset{a}{A}_\mu \overset{a}{F} - K_{,\bar{\alpha}}\, D_\mu z^{\bar{\alpha}} - \overset{a}{A}_\mu \overset{a}{\bar{F}}\Big) = \frac{1}{2i}\Big(K_{,\alpha}\,\partial_\mu z^\alpha - K_{,\bar{\alpha}}\,\partial_\mu z^{\bar{\alpha}}\Big) + \overset{a}{A}_\mu \overset{a}{D} \ . \tag{2.8}$$

It couples with gravitational strength and thus appears with coefficient κ^2 which is set to $\kappa = 1$ in the present notation. Under the gauge transformation

$$\delta z^\alpha = \overset{a}{\theta}\,\overset{a}{X}{}^\alpha \ , \quad \delta z^{\bar{\beta}} = \overset{a}{\theta}\,\overset{a}{X}{}^{\bar{\beta}} \ , \quad \delta \overset{a}{A}_\mu \ = \ \partial_\mu \overset{a}{\theta} + f^{abc}\overset{b}{A}_\mu \overset{c}{\theta} \ , \tag{2.9}$$

one may show that B_μ transforms as an abelian gauge connection, that is

$$\delta B_\mu = \partial_\mu(\overset{a}{\theta}\,\mathrm{Im}(\overset{a}{F})) \ . \tag{2.10}$$

The mathematical background just described leads to the bosonic Lagrangian of the non-linear σ-model on $\mathcal{T}$

$$\mathcal{L}_b = -G_{\alpha\bar{\beta}} D_\mu z^\alpha D^\mu z^{\bar{\beta}} \tag{2.11}$$

with covariant derivatives

$$D_\mu z^\alpha \ = \ \partial_\mu z^\alpha - \overset{a}{A}_\mu \overset{a}{X}{}^\alpha \ , \quad D_\mu z^{\bar{\beta}} \ = \ \partial_\mu z^{\bar{\beta}} - \overset{a}{A}_\mu \overset{a}{X}{}^{\bar{\beta}} \ . \tag{2.12}$$

The supersymmetric partners of the bosons are Weyl spinors which transform as tangent vectors under holomorphic diffeomorphisms of $\mathcal{T}$. We write these as 4-component spinors with chiral projectors $L, R = (1 \mp \gamma_5)/2$. Thus a chiral multiplet consists of the set $(z^\alpha, L\psi^\alpha)$, while an anti-chiral multiplet is $(z^{\bar{\beta}}, \bar{\psi}^{\bar{\beta}} R)$. The fermion kinetic Lagrangian is

$$\mathcal{L}_f \ = \ G_{\alpha\bar{\beta}}\bar{\psi}^{\bar{\beta}}\gamma^\mu D_\mu L\psi^\alpha \ = \ - G_{\alpha\bar{\beta}}(D_\mu \bar{\psi}^{\bar{\beta}} R)\gamma^\mu \psi^\alpha \tag{2.13}$$

with covariant derivatives[3]

$$\begin{aligned} D_\mu L\psi^\alpha &= (\partial_\mu \delta^\alpha_\beta + \Gamma^\alpha_{\beta\gamma}\partial_\mu z^\gamma - \overset{a}{A}_\mu \overset{a}{X}{}^\alpha{}_{;\beta})L\psi^\beta \ , \\ D_\mu \bar\psi^{\bar\beta} R &= (\partial_\mu \delta^{\bar\beta}_{\bar\gamma} + \Gamma^{\bar\beta}_{\bar\gamma\bar\delta}\partial_\mu \bar z^{\bar\delta} - \overset{a}{A}_\mu \overset{a}{X}{}^{\bar\beta}{}_{;\bar\gamma})\bar\psi^{\bar\gamma} R \ . \end{aligned} \tag{2.14}$$

It is a useful exercise to show that under the gauge transformations of (2.9) for the bosons and

$$\delta L\psi^\alpha = \overset{a}{\theta}\, \overset{a}{X}{}^\alpha{}_{,\beta}\, L\psi^\beta \ , \tag{2.15}$$

for fermions the covariant derivatives transform as (holomorphic components of) tangent vectors, i.e.

$$\delta D_\mu z^\alpha = \overset{a}{\theta}\, \overset{a}{X}{}^\alpha{}_{,\beta}\, D_\mu z^\beta \ , \quad \delta D_\mu L\psi^\alpha = \overset{a}{\theta}\, \overset{a}{X}{}^\alpha{}_{,\beta}\, D_\mu L\psi^\beta \ . \tag{2.16}$$

3 Anomalies and Inconsistency

In this section we will derive the consistency condition in a succession of models which gradually incorporate the features of the full gauged supergravity models which we tackle in section 4. It is worth stating our strategy in general terms before embarking on the detailed analysis.

3.1 The strategy

The models considered include gauge fields A^a_μ, scalars $z^\alpha,\ z^{\bar\alpha}$ of a Kähler σ-model, and fermions (the gravitino, gauginos and chiral fermions) which we temporarily denote by ψ. The gauge current in all models is defined by

$$\overset{a}{J}{}^\rho \equiv -\frac{\delta(\mathcal{L} + \frac14 F^b_{\mu\nu}F^{b\mu\nu})}{\delta A^{a\rho}} = \overset{a}{J}{}^\rho_b + \overset{a}{J}{}^\rho_f \tag{3.1}$$

and is the sum of a purely bosonic term $J^{a\nu}_b$ and a term $J^{a\nu}_f$ involving the fermions. The gauge field satisfies the Yang-Mills equation (1.2) and the consistency condition (1.3) requires conservation of the full current, i.e.

$$D_\nu(\overset{a}{J}{}^\nu_b + \overset{a}{J}{}^\nu_f) = 0 \ . \tag{3.2}$$

[3] The Kähler connection B_μ will be included in later sections. The Lagrangian of the supersymmetric σ-model [10] includes another term $\frac14 R_{\alpha\bar\beta\gamma\bar\delta}(\bar\psi^\alpha L\psi^\gamma)(\bar\psi^{\bar\beta} R\psi^{\bar\delta})$ and a Yukawa coupling to gauginos. We will discuss this quartic term in footnote 5, and the Yukawa coupling will be included in the supergravity analysis.

One way to isolate the anomalous fermion terms in the divergence of the current is to use the scalar equations of motion (which include fermion terms) to evaluate $D_\nu J^{a\nu}$ in (3.2). However, this process is rather complicated in the context of the gauged σ-model, and it can be bypassed as we now describe.

In any gauge invariant model there is a functional identity which expresses the gauge invariance at the *classical* level, namely

$$\overset{a}{\theta}(x)\Big[D_\nu \overset{a}{J}{}^\nu + \overset{a}{X}{}^\alpha \frac{\delta\mathcal{L}}{\delta z^\alpha} + \overset{a}{X}{}^{\bar\beta}\frac{\delta\mathcal{L}}{\delta z^{\bar\beta}} + \overset{a}{\delta}\,\bar\psi\frac{\delta\mathcal{L}}{\delta\bar\psi}\Big] \;\equiv\; 0 \;, \tag{3.3}$$

in which the gauge variations of the bosons of (2.9) appear and gauge variations of all the fermions are symbolically denoted by $\delta^a\psi$. As is very well known, this gauge identity tells us that, no matter how complicated the model, $D_\nu\, J^{a\nu}$ vanishes *classically* if *all* charged fields satisfy their equations of motion. It may also be interpreted as the statement that, if the *scalar* equations of motion are satisfied, the functional form of $D_\nu\, J^{a\nu}$ is the negative of the fermion gauge variation of the Lagrangian. The consistency condition (1.3) may be viewed as a cancellation condition for the correction to the gauge identity due to anomalies.

3.2 A model with gauginos

We now apply the strategy just outlined to derive the consistency condition for a model which incorporates some features of gauged supergravity theories. The model contains a (non-abelian) gauge field coupled to complex scalars z^α which determine a gauged non-linear σ-model whose target space is the n-dimensional Kähler manifold $\mathcal{T}$ and to Majorana fermions λ^a in the adjoint representation of the gauged isometry group G. Since we leave out the chiral fermions, the superpartners of the z^α, this model is not supersymmetric. Its Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\overset{a}{F}_{\mu\nu}\overset{a}{F}{}^{\mu\nu} - G_{\alpha\bar\beta}D_\mu z^\alpha D^\mu z^{\bar\beta} + \frac{1}{2}\overset{a}{\bar\lambda}\gamma^\mu D_\mu \overset{a}{\lambda} \;. \tag{3.4}$$

The boson covariant derivatives are given in (2.12), while the fermion covariant derivative

$$D_\mu \overset{a}{\lambda} = \partial_\mu \overset{a}{\lambda} + f^{abc}\overset{b}{A}_\mu \overset{c}{\lambda} + \frac{1}{2} i B_\mu \gamma_5 \overset{a}{\lambda} \tag{3.5}$$

includes both the expected elementary connection and the composite Kähler connection B_μ defined in (2.8). We include B_μ in the present non-gravitational model to illustrate the important role it plays in the anomaly analysis.

The Lagrangian (3.4) is gauge invariant under the transformations of (2.15) for bosonic fields and

$$\delta \overset{a}{\lambda} = f^{abc} \overset{b}{\lambda} \overset{c}{\theta} - \frac{1}{2} i \overset{b}{\theta} \operatorname{Im}(\overset{b}{F}) \gamma_5 \overset{a}{\lambda} \tag{3.6}$$

for the fermions. It also has the global axial symmetry $\delta\lambda^a = i\alpha\gamma_5\lambda^a$ with Noether current

$$N^\mu = -\frac{i}{2} \overset{a}{\bar{\lambda}} \gamma^\mu \gamma_5 \overset{a}{\lambda} \ . \tag{3.7}$$

Although we do not need the explicit form of the equations of motion, we record them for completeness. One can write the gauge field equation as

$$D_\mu \overset{a}{F}{}^{\mu\nu} = \overset{a}{J}{}^\nu \equiv -\frac{\delta(\mathcal{L} + \frac{1}{4} F^b_{\rho\sigma} F^{b\rho\sigma})}{\delta A^a_\nu} = -G_{AB} \overset{a}{X}{}^A D^\nu z^B + \overset{a}{j}{}^\nu + \frac{1}{2} \overset{a}{D} N^\nu \tag{3.8}$$

with the adjoint gauge current $j^{a\nu} = \frac{1}{2} f^{abc} \bar{\lambda}^b \gamma^\nu \lambda^c$. The equation for the scalar z^α is

$$D^\nu D_\nu z^\alpha \ \ = \ \ -\frac{1}{2} i \Big(N^\nu D_\nu z^\alpha + \frac{1}{2} G^{\alpha\bar{\beta}} K_{,\bar{\beta}} \, \partial_\nu N^\nu \Big) \ . \tag{3.9}$$

The left sides of these equations contain the gauge covariant harmonic map operator $D^\nu D_\nu z^\alpha \equiv (\partial^\nu \delta^\alpha_\beta + \Gamma^\alpha_{\beta\gamma} \partial^\nu z^\gamma - A^{a\nu} \, X^{a\alpha}{}_{;\beta}) D_\nu z^\beta$. The equation for $z^{\bar{\alpha}}$ is the complex conjugate of (3.9).

To derive the consistency condition we insert the Dirac conjugate of the fermion gauge variation (3.6) in the gauge identity (3.3), assuming, of course, that (3.9) is satisfied. After some index shuffling, we obtain

$$\overset{a}{\theta} D_\nu \overset{a}{J}{}^\nu = \overset{a}{\theta} \Big[f^{abc} \overset{b}{\bar{\lambda}} \gamma^\mu D_\mu \overset{c}{\lambda} + \frac{1}{2} i \operatorname{Im}(\overset{a}{F}) \overset{b}{\bar{\lambda}} \gamma_5 \gamma^\mu D_\mu \overset{b}{\lambda} \Big]. \tag{3.10}$$

Note that the B_μ connection cancels in $D_\mu \lambda^b$ due to the symmetry properties of Majorana bilinears. The Majorana properties also allow us to extract D_μ as a total gauge covariant derivative. Then dropping the overall factor of θ^a, we can express the consistency condition as

$$0 = D_\nu \overset{a}{J}{}^\nu = D_\nu \overset{a}{j}{}^\nu + \frac{1}{2} \operatorname{Im}(\overset{a}{F}) \partial_\nu N^\nu \ . \tag{3.11}$$

The currents N^ν and $j^{a\nu}$ are conserved classically, so we would obtain the identity $0 = 0$ reflecting the classical gauge invariance of the theory. The advantage of the consistency condition (3.11) is that it enables us to bring in the quantum level violation of gauge invariance due to the anomaly.

3.3 The axial anomaly with gauginos

The anomalous contribution of (3.11) may be calculated by the method of Fujikawa in which we consider the change in the fermion measure in the path integral due to the changes of variable $\delta\lambda^a = i\alpha(x)\gamma_5\lambda^a$ and $\delta\lambda^a = f^{abc}\lambda^b\theta^c(x)$. In order to minimize repetition of well known material we will give only brief discussion and refer readers to the monograph of Fujikawa and Suzuki [5]. Results for the several cases of anomalies needed in this paper can be obtained by modification of the appropriate sections of [5]. All results are stated in terms of the *covariant* anomaly.

In particular the anomalous divergence $\partial_\nu N^\nu$ in our model may be obtained by rewriting the Majorana kinetic action in terms of Weyl spinors and using sections 6.4 and 6.4.1 of [5]. Fermion mode functions are defined to satisfy

$$\not{D}_R \not{D}_L \overset{a}{\phi}_n = -\rho_n^2 \overset{a}{\phi}_n \tag{3.12}$$

with

$$\begin{aligned} \overset{ab}{\not{D}}_L &= \gamma^\mu \overset{ab}{D}_\mu L = \gamma^\mu \Big[(\partial_\mu - \frac{1}{2}iB_\mu)\delta^{ab} - f^{abc} \overset{c}{A}_\mu\Big] L \ , \\ \overset{ab}{\not{D}}_R &= -\overset{ab}{\not{D}}{}_L^\dagger = \gamma^\mu \Big[(\partial_\mu - \frac{1}{2}iB_\mu)\delta^{ab} - f^{abc} \overset{c}{A}_\mu\Big] R \ . \end{aligned} \tag{3.13}$$

Note that

$$\not{D}_R \not{D}_L = \Big[D^\mu D_\mu + \frac{1}{2}\gamma^{\mu\nu}\mathcal{F}_{\mu\nu}\Big] L \ , \tag{3.14}$$

in which $\mathcal{F}^{ab}_{\mu\nu}$ is the field strength of the full connection in (3.13), namely

$$\begin{aligned} \overset{ab}{\mathcal{F}}_{\mu\nu} &= -f^{abc} \overset{c}{F}_{\mu\nu} - \frac{1}{2}iB_{\mu\nu}\delta^{ab} \ , \\ \overset{c}{F}_{\mu\nu} &= \partial_\mu \overset{c}{A}_\nu - \partial_\nu \overset{c}{A}_\mu + f^{cde} \overset{d}{A}_\mu \overset{e}{A}_\nu \ , \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ . \end{aligned} \tag{3.15}$$

The anomaly of the Noether current N_μ of $\delta\lambda^a = i\alpha(x)\gamma_5\lambda^a$ is

$$\langle \partial_\nu N^\nu \rangle = -\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} \overset{ab}{\mathcal{F}}_{\mu\nu} \overset{ba}{\mathcal{F}}_{\rho\sigma} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\Big[C_2(G) \overset{b}{F}_{\mu\nu} \overset{b}{F}_{\rho\sigma} + \frac{1}{4}n_\lambda B_{\mu\nu}B_{\rho\sigma}\Big] \ , \tag{3.16}$$

where $n_\lambda = \dim(G)$ is the total number of gauginos and $C_2(G)\delta^{ab} = f^{acd}f^{bcd}$ is the adjoint Casimir operator. The vector current is also anomalous here because it couples to the B_μ connection. Its anomaly is

$$\langle D_\nu \overset{a}{j}{}^\nu \rangle = \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} f^{abc} \overset{cd}{\mathcal{F}}_{\mu\nu} \overset{db}{\mathcal{F}}_{\rho\sigma} = \frac{1}{32\pi^2} C_2(G)\epsilon^{\mu\nu\rho\sigma} \overset{a}{F}_{\mu\nu} B_{\rho\sigma} \ . \tag{3.17}$$

Combining (3.16,3.17), and (3.11), we find that the "inconsistency condition" of the model reads

$$0 \;=\; \frac{1}{2}\mathrm{Im}(\overset{a}{F}\,)\,\epsilon^{\mu\nu\rho\sigma}\Big[C_2(G)\,\overset{b}{F}_{\mu\nu}\,\overset{b}{F}_{\rho\sigma}+\frac{1}{4}n_\lambda B_{\mu\nu}B_{\rho\sigma}\Big]+C_2(G)\epsilon^{\mu\nu\rho\sigma}\,\overset{a}{F}_{\mu\nu}B_{\rho\sigma}\;. \qquad (3.18)$$

The model appears to be fatally inconsistent since the coefficients of all three terms on the right-hand-side have to vanish. We will discuss in [6] the possibility that parts of the anomaly may be removed by adding local non-gauge invariant polynomials in the gauge potential and the Kähler connection to the classical action. However, it will turn out that no such counter terms exist to remove the term involving $B_{\mu\nu}B_{\rho\sigma}$. Therefore, the model is indeed inconsistent unless the Kähler potential is gauge invariant.

3.4 Models with chiral fermions

We now analyze another model in similar fashion, a model in which the non-abelian gauge field is coupled both to the bosons and fermions of the Kähler manifold σ-model. The features of both models discussed in this section, and more, will be combined to treat the general gauged supergravity model in section 4. The Lagrangian of this model is

$$\mathcal{L}=-\frac{1}{4}\,\overset{a}{F}_{\mu\nu}\,\overset{a}{F}{}^{\mu\nu}-G_{\alpha\bar\beta}D_\mu z^\alpha D^\mu z^{\bar\beta}+G_{\alpha\bar\beta}\bar\psi^{\bar\beta}\gamma^\mu D_\mu L\psi^\alpha \qquad (3.19)$$

with fermion covariant derivative[4]

$$D_\mu L\psi^\alpha=\Big(\partial_\mu\delta^\alpha_\beta+\Gamma^\alpha_{\beta\gamma}\partial_\mu z^\gamma-\overset{a}{A}_\mu\,\overset{a}{X}{}^\alpha{}_{;\beta}+\frac{1}{2}iB_\mu\delta^\alpha_\beta\Big)L\psi^\beta\;. \qquad (3.20)$$

The Lagrangian is gauge invariant if the boson transformations of (2.15) are combined with

$$\delta L\psi^\alpha=\overset{a}{\theta}\Big(\,\overset{a}{X}{}^\alpha{}_{,\beta}\,L\psi^\beta-\frac{i}{2}\mathrm{Im}(\overset{a}{F}\,)L\psi^\alpha\Big)\;, \qquad (3.21)$$

and there is a global axial symmetry $\delta L\psi^\alpha=i\alpha L\psi^\alpha$ with Noether current

$$N^\mu=-iG_{\alpha\bar\beta}\bar\psi^{\bar\beta}\gamma^\mu L\psi^\alpha\;. \qquad (3.22)$$

The gauge field equation is

$$\begin{aligned}D_\mu\,\overset{a}{F}{}^{\mu\nu}\;&=\;\overset{a}{J}{}^\nu\;\equiv\;-\frac{\delta(\mathcal{L}+\frac{1}{4}F^b_{\rho\sigma}F^{b\rho\sigma})}{\delta A^a_\nu}\;=\;\overset{a}{J}{}^\nu_b+\overset{a}{J}{}^\nu_f+\frac{1}{2}\,\overset{a}{D}\,N^\nu\;,\\ \overset{a}{J}{}^\nu_b\;&=\;-G_{\alpha\bar\beta}\Big(\,\overset{a}{X}{}^\alpha D^\nu z^{\bar\beta}+\overset{a}{X}{}^{\bar\beta}D^\nu z^\alpha\Big)\;,\quad \overset{a}{J}{}^\nu_f\;=\;\overset{a}{X}_{\bar\beta;\alpha}\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha\;,\end{aligned} \qquad (3.23)$$

[4]Note that B_μ couples to $L\psi^\alpha$ and $L\lambda^a$ with opposite sign.

and the scalar equations are

$$\frac{\delta\mathcal{L}}{\delta z^{\bar\gamma}} \;=\; 0 \;=\; G_{\alpha\bar\gamma}D_\mu D^\mu z^\alpha + R_{\delta\bar\beta\alpha\bar\gamma}D_\nu z^\delta(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha) \qquad (3.24)$$
$$+G_{\alpha\bar\beta,\bar\gamma}\bar\psi^{\bar\beta}\gamma^\mu D_\mu L\psi^\alpha + \frac{i}{2}G_{\delta\bar\gamma}D_\mu z^\delta N^\mu + \frac{i}{4}K_{,\bar\gamma}\,\partial_\mu N^\mu \;,$$
$$\frac{\delta\mathcal{L}}{\delta z^{\gamma}} \;=\; 0 \;=\; G_{\gamma\bar\beta}D_\mu D^\mu z^{\bar\beta} - R_{\bar\delta\alpha\bar\beta\gamma}D_\nu z^{\bar\delta}(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha) \qquad (3.25)$$
$$+G_{\alpha\bar\beta,\gamma}\bar\psi^{\bar\beta}\gamma^\mu D_\mu L\psi^\alpha - \frac{i}{2}G_{\gamma\bar\beta}D_\mu z^{\bar\beta}N^\mu - \frac{i}{4}K_{,\gamma}\,\partial_\mu N^\mu - G_{\delta\bar\beta}\Gamma^\delta_{\alpha\gamma}D_\mu(\bar\psi^{\bar\beta}\gamma^\mu L\psi^\alpha)\;.$$

Note that $R_{\bar\gamma\beta\bar\alpha\delta} = R_{\bar\gamma\delta\bar\alpha\beta}$. The second equation becomes the conjugate of the first after applying the covariant divergence to the fermions in the last term of (3.25).

It is a very intricate task (which we have done) to derive the consistency condition for this model by combining equations of motion. The complicated scalar field equations are indicative of the difficulty. It is far simpler to use the gauge identity (3.3). For this we simply need the fermion gauge variation of the Lagrangian, using both $\delta L\psi^\alpha$ in (3.21) and its conjugate $\delta\bar\psi^{\bar\beta}R$. After dropping the common factor θ^a, the gauge identity can be written immediately as the consistency condition[5]

$$D_\nu \overset{a}{J}{}^\nu \;= \qquad (3.26)$$
$$-G_{\alpha\bar\beta}\Big[\Big(\overset{a}{X}{}^{\bar\beta}{}_{,\bar\gamma} + \frac{i}{2}\mathrm{Im}(\overset{a}{F})\delta^{\bar\beta}_{\bar\gamma}\Big)\bar\psi^{\bar\gamma}\gamma^\mu D_\mu L\psi^\alpha - \Big(\overset{a}{X}{}^{\alpha}{}_{,\gamma} - \frac{i}{2}\mathrm{Im}(\overset{a}{F})\delta^{\alpha}_{\gamma}\Big)(D_\mu\bar\psi^{\bar\beta}R)\gamma^\mu\psi^\gamma\Big]\;.$$

The non-covariant terms in this expression originate in the fact that σ-model fermions transform under gauge variations as tangent vectors on the target space and thus carry the non-covariant factor $X^{a\alpha}{}_{,\beta}$. The result (3.26) may be rearranged to read

$$0 \;=\; D_\nu \overset{a}{J}{}^\nu \;=\; \frac{1}{2}\Big(G_{\gamma\bar\beta}\overset{a}{X}{}^{\gamma}{}_{,\alpha} - G_{\alpha\bar\gamma}\overset{a}{X}{}^{\bar\gamma}{}_{,\bar\beta}\Big)D_\nu(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha) + \frac{1}{2}\mathrm{Im}(\overset{a}{F})\partial_\nu N^\nu \qquad (3.27)$$
$$-\frac{1}{2}\Big(G_{\gamma\bar\beta}\overset{a}{X}{}^{\gamma}{}_{,\alpha} + G_{\alpha\bar\gamma}\overset{a}{X}{}^{\bar\gamma}{}_{,\bar\beta}\Big)(\bar\psi^{\bar\beta}\gamma^\mu D_\mu L\psi^\alpha - (D_\mu\bar\psi^{\bar\beta})\gamma^\mu L\psi^\alpha)\;.$$

We are again in a situation in which the consistency condition would be satisfied if the classical fermion field equations are used, but the form of the right side allows us to probe possible quantum anomalies. Indeed, the anomalies of $D_\nu(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha)$ and $\partial_\nu N^\nu$ will be obtained in the next section. One might suspect an anomalous one-loop contribution from the fermion bilinear in the last term of (3.27) because this term

[5] Suppose that we modify the model by adding the quartic $R\psi^4$ term in footnote 3 that is part of the supersymmetric non-linear σ-model and repeat the analysis. The consistency condition would be modified by the fermion gauge variation of the quartic term. We will assume that this does not modify the anomalies and leave this complication aside.

resembles the trace of a fermion stress tensor. However, we have checked that the contributing Feynman diagrams conform to the trace anomaly calculation of [11] and cancel.[6]

3.5 The axial anomaly with chiral fermions

The first two of the three terms of (3.27) contain quantum anomalies whose calculation can be modeled on that of section 6.4 of [5]. To adapt the treatment of section 6.4 to the situation of the non-linear σ-model we introduce frames on the target space $\mathcal{T}$ via

$$ds^2 = 2G_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}} = 2\delta_{i\bar{\jmath}}e^i_\alpha dz^\alpha e^{\bar{\jmath}}_{\bar{\beta}}dz^{\bar{\beta}} \tag{3.28}$$

and use the frame basis to reexpress the σ-model fermions as

$$L\psi^\alpha = e^\alpha_i L\psi^i\ , \quad \bar{\psi}^{\bar{\beta}}R = e^{\bar{\beta}}_{\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}R\ , \tag{3.29}$$

where e^α_i, $e^{\bar{\beta}}_{\bar{\jmath}}$ are inverse frames. The fermion kinetic Lagrangian of our model can be rewritten in the frame basis as

$$G_{\alpha\bar{\beta}}\bar{\psi}^{\bar{\beta}}\gamma^\mu D_\mu L\psi^\alpha = \bar{\psi}_i\gamma^\mu D_\mu L\psi^i \tag{3.30}$$

with

$$\begin{aligned}
\bar{\psi}_i R &= \delta_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}R\ , \quad \overset{a}{X}{}^i{}_{;j} = e^i_\alpha \overset{a}{X}{}^\alpha{}_{;\beta}\, e^\beta_j\ , \quad \Omega^i_{Cj} = e^i_\beta e^\beta_{j,C} + e^i_\alpha\Gamma^\alpha_{\beta C}e^\beta_j\ , \\
D^i_{\mu j}L\psi^j &= \Big[(\partial_\mu + \frac{1}{2}iB_\mu)\delta^i_j + \Omega^i_{Cj}\partial_\mu z^C - \overset{a}{A}_\mu \overset{a}{X}{}^i{}_{;j}\Big]L\psi^j\ .
\end{aligned} \tag{3.31}$$

The $2n$-valued index C appears because the Kähler spin connection couples to both $\partial_\mu z^\gamma$ and $\partial_\mu z^{\bar{\gamma}}$.

As discussed in [13], the fermions ψ^i are sections of a holomorphic vector bundle on $\mathcal{T}$ with structure group $U(n)$. The new fermion kinetic term (3.30) is separately invariant under diffeomorphisms of $\mathcal{T}$ and unitary transformations of the frame and fermion fields, viz.

$$\begin{aligned}
e^i_\alpha &\rightarrow U^i_j e^j_\alpha\ , \quad e_{i\bar{\alpha}} = \delta_{i\bar{\imath}}e^{\bar{\imath}}_{\bar{\alpha}} \rightarrow e_{j\bar{\alpha}}U^{\dagger j}_i\ , \\
\psi^i &\rightarrow U^i_j\psi^j\ , \quad \bar{\psi}_i \rightarrow \bar{\psi}_j U^{\dagger j}_i\ ,
\end{aligned} \tag{3.32}$$

[6] This is also described in section 19.5 and figure 19.10 of [12]. The trace anomaly in Yang-Mills theory comes from another diagram, not present in our case, involving the gauge field stress tensor and fermion vacuum polarization.

where $U^i_j(z,\bar z)$ is a unitary matrix. This means that we can apply the discussion of section 6.4 of [5] quite directly and obtain the anomalous response to the transformation (3.32) of the fermion fields in the path integral measure. For this purpose we introduce a standard basis of generators T^{ai}_j, with $a = 0, 1, \ldots, n^2-1$, of the fundamental representation of $U(n)$, normalized by $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$, and write $U = \exp(i\alpha^aT^a)$. The $U(1)$ generator is $T^{0i}_j = \delta^i_j/\sqrt{2n}$.

Following [5], we expand the field ψ^i in mode functions ϕ^i_n which satisfy (3.12), but with the new operator

$$(\not{D}_L)^i_j = \gamma^\mu D^i_{\mu j} L \tag{3.33}$$

and $D^i_{\mu j}$ given in (3.31). The relation (3.14) holds with field strength (see also [14])

$$\begin{aligned}\mathcal{F}^i_{\mu\nu j} &= R_{AB}{}^i{}_j D_\mu z^A D_\nu z^B - \overset{a}{F}_{\mu\nu}\, \overset{a}{X}{}^i{}_{;j} + \frac{1}{2} i B_{\mu\nu}\delta^i_j\ ,\\ R_{AB}{}^i{}_j &= \Omega^i_{Aj,B} + \Omega^i_{Al}\Omega^l_{Bj} - \Omega^i_{Bj,A} - \Omega^i_{Bl}\Omega^l_{Aj}\ .\end{aligned} \tag{3.34}$$

It then follows from sections 6.4.1 and 6.4.2 of [5] that the anomalous Jacobian $J(\alpha^aT^a)$ of the path integral measure is

$$\ln J(\overset{a}{\alpha}\,\overset{a}{T}) = \frac{i}{32\pi^2}\int d^4x\ \overset{a}{\alpha}(x)\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(\overset{a}{T}\,\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma})\ . \tag{3.35}$$

The anomalous conservation law of the $U(n)$ current is

$$\langle D_\nu(\bar\psi_i \overset{a}{T}{}^i_j\gamma^\mu L\psi^j)\rangle = \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(\overset{a}{T}\,\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma})\ . \tag{3.36}$$

We now relate this result to the anomalous divergence in (3.27) by expressing that current in the frame basis and using the completeness relation of the matrices T^a, namely

$$\overset{a}{T}{}^i_j\,\overset{a}{T}{}^k_l = \delta^i_l\delta^k_j\ , \tag{3.37}$$

to write

$$\begin{aligned}\langle D_\nu(\bar\psi_\beta\gamma^\nu L\psi^\alpha)\rangle &= e^j_\beta e^\alpha_i\,\overset{a}{T}{}^i_j\langle D_\nu(\bar\psi\,\overset{a}{T}\,\gamma^\nu\psi)\rangle\\ &= \frac{i}{32\pi^2}e^j_\beta e^\alpha_i\,\overset{a}{T}{}^i_j\,\overset{a}{T}{}^k_l\,\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^l_{\mu\nu m}\mathcal{F}^m_{\rho\sigma k}\\ &= \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\alpha_{\mu\nu\gamma}\mathcal{F}^\gamma_{\rho\sigma\beta}\ .\end{aligned} \tag{3.38}$$

In the last line we converted the field strength to the coordinate basis of $\mathcal{T}$ in which

$$\mathcal{F}^\alpha_{\mu\nu\beta} = R_{\gamma\bar\delta}{}^\alpha{}_\beta(D_\mu z^\gamma D_\nu z^{\bar\delta} - D_\nu z^\gamma D_\mu z^{\bar\delta}) - \overset{a}{F}_{\mu\nu}\,\overset{a}{X}{}^\alpha{}_{;\beta} + \frac{i}{2}B_{\mu\nu}\delta^\alpha_\beta\ . \tag{3.39}$$

The inconsistency condition then reads

$$0 = \langle D_\nu \overset{a}{J}{}^\nu \rangle = -\frac{1}{32\pi^2}\Big[G^{\beta\bar\gamma}\overset{a}{Y}_{\alpha\bar\gamma} - \frac{1}{2}\mathrm{Im}(\overset{a}{F})\delta_\alpha^\beta\Big]\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\alpha_{\mu\nu\gamma}\mathcal{F}^\gamma_{\rho\sigma\beta}\ , \tag{3.40}$$

in which we use the abbreviation

$$\overset{a}{Y}_{\alpha\bar\beta} = \frac{1}{2i}\Big(G_{\gamma\bar\beta}\overset{a}{X}{}^\gamma{}_{,\alpha} - G_{\alpha\bar\gamma}\overset{a}{X}{}^{\bar\gamma}{}_{,\bar\beta}\Big)\ . \tag{3.41}$$

When the field strength (3.39) is inserted in (3.40) one finds a rather complex but correct expression for the anomaly. Even when $\mathrm{Im}(F^a) = 0$ the anomaly contains new terms due to the presence of $B_{\mu\nu}$ in (3.39).

3.6 An example: $\mathbb{CP}^1$

It is useful to treat a specific example which has the features of the general σ-model discussed in section 3.3, yet is simple enough that one can obtain the consistency condition without the full geometrical baggage. Therefore we outline briefly the case of the target space $\mathbb{CP}^1$. The isometry group is $SU(2)$ with three holomorphic Killing vectors $X^{a\alpha}(z)$, $a = 1, 2, 3$. The Kähler potential, however, is not invariant under $SU(2)$. Thus $\mathrm{Im}(F^a(z)) \neq 0$, and Kähler anomalies play a role in the consistency conditions. To analyze this model we need the Kähler potential, metric, and connection

$$K = \ln(1 + z\bar z)\ , \quad G_{z\bar z} = (1 + z\bar z)^{-2}\ , \quad \Gamma^z_{zz} = -2\bar z(1 + z\bar z)^{-1}\ , \tag{3.42}$$

and the Killing vectors and D-terms

$$\begin{aligned}
\overset{1}{X}{}^z &= -\frac{i}{2}(1 - z^2)\ , & \overset{1}{D} &= \frac{1}{2}\frac{z + \bar z}{1 + z\bar z}\ ,\\
\overset{2}{X}{}^z &= \frac{1}{2}(1 + z^2)\ , & \overset{2}{D} &= -\frac{i}{2}\frac{z - \bar z}{1 + z\bar z}\ ,\\
\overset{3}{X}{}^z &= -iz\ , & \overset{3}{D} &= -\frac{1}{2}\frac{1 - z\bar z}{1 + z\bar z}\ .
\end{aligned} \tag{3.43}$$

This leads to

$$\overset{1}{F} = \frac{i}{2}z\ , \quad \overset{2}{F} = \frac{1}{2}z\ , \quad \overset{3}{F} = -\frac{i}{2}\ . \tag{3.44}$$

Note that the Kähler potential is in fact invariant under the third isometry X^{3z}, but still $\mathrm{Im}(F^3) \neq 0$. Due to (2.5) it is fixed to the non-vanishing constant value above.

We consider the $\mathbb{CP}^1$ σ-model, with a chiral fermion, and with the full $SU(2)$ symmetry group gauged. However, to simplify the equations, we set the gauge fields

A^1_μ and A^2_μ to zero[7] and focus on the inconsistency related to X^{3z}. Similar results would be obtained for the other two isometries. We also leave out the gauginos. In this model the Lagrangian (3.19) becomes:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\overset{3}{F}_{\mu\nu}\overset{3}{F}{}^{\mu\nu} - \frac{1}{(1+z\bar{z})^2}\Big[(\partial_\mu - i\overset{3}{A}_\mu)\bar{z}(\partial^\mu + i\overset{3}{A}{}^\mu)z \\
&\qquad\qquad -\bar{\psi}\gamma^\mu\Big(\partial_\mu - \frac{2\bar{z}\partial_\mu z}{1+z\bar{z}} + i\frac{1-z\bar{z}}{1+z\bar{z}}\overset{3}{A}_\mu + \frac{1}{2}iB_\mu\Big)L\psi\Big] , \\
B_\mu &= \frac{\mathrm{Im}(\bar{z}\partial_\mu z) - \frac{1}{2}(1-z\bar{z})A^3_\mu}{1+z\bar{z}} .
\end{aligned} \tag{3.45}
$$

It is now straightforward to obtain by direct calculation, ignoring the geometrical origin of the terms in (3.45), the consistency condition

$$
\begin{aligned}
0 &= -D_\mu\frac{\delta\mathcal{L}}{\delta A^3_\mu} - iz\frac{\delta\mathcal{L}}{\delta z} + i\bar{z}\frac{\delta\mathcal{L}}{\delta\bar{z}} = \\
&\qquad -\frac{i}{(1+z\bar{z})^2}D_\nu(\bar{\psi}\gamma^\nu L\psi) - \frac{1}{4}\partial_\nu\Big[-\frac{i}{(1+z\bar{z})^2}\bar{\psi}\gamma^\nu L\psi\Big] ,
\end{aligned} \tag{3.46}
$$

where D_μ is the full σ-model covariant derivative. This result can be compared with the general expression (3.40). The first term reproduces $Y^3_{\alpha\bar{\beta}} = -G_{\alpha\bar{\beta}}$, the second $\mathrm{Im}(F^3) = -1/2$. In this simple model the two terms in the last line of (3.46) are proportional, so we get

$$
0 = -\frac{3}{4}\partial_\nu N^\nu , \tag{3.47}
$$

as a special case of (3.40), with the $U(1)$ axial current

$$
N^\nu = -\frac{i}{(1+z\bar{z})^2}\bar{\psi}\gamma^\nu L\psi . \tag{3.48}
$$

The axial anomaly is

$$
\langle\partial_\nu N^\nu\rangle = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} , \tag{3.49}
$$

in which $\mathcal{F}_{\mu\nu}$ is the field strength of the connection in (3.45), namely

$$
\mathcal{F}_{\mu\nu} = \frac{2}{(1+z\bar{z})^2}(D_\mu z D_\nu\bar{z} - D_\nu z D_\mu\bar{z}) + i\frac{1-z\bar{z}}{1+z\bar{z}}\overset{3}{F}_{\mu\nu} + \frac{1}{2}iB_{\mu\nu} . \tag{3.50}
$$

This agrees with (3.39) in the $\mathbb{CP}^1$ model.

[7] If we were to gauge only the $U(1)$ referring to X^{3z} then the constant $\mathrm{Im}(F^3)$ would be arbitrary and interpreted as the Fayet-Iliopoulos coupling. In particular we would have the freedom to set $F^3 = 0$, since the Kähler potential is invariant.

4 The general gauged supergravity model

The supersymmetric σ-model coupled to supergravity includes the gravitino and various coupling terms in addition to the terms studied in the previous sections. The action is

$$\mathcal{S}[e^i_\mu, \overset{a}{A}_\mu, z^\alpha, z^{\bar\beta}, \psi^\alpha, \bar\psi^{\bar\beta}, \overset{a}{\lambda}, \Psi_\mu] = \int d^4x \det(e^i_\mu)\mathcal{L}_{\rm SG} \ , \tag{4.1}$$

with Lagrangian density (in conventions similar to those of [15])

$$\mathcal{L}_{\rm SG} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\rm int} + \text{ quartic terms} \tag{4.2}$$

with

$$\begin{aligned}
\mathcal{L}_b &= \frac{1}{2}R - \frac{1}{4}\overset{a}{F}_{\mu\nu}\overset{a}{F}{}^{\mu\nu} - \frac{1}{2}\overset{a}{D}\overset{a}{D} - G_{\alpha\bar\beta}D_\mu z^\alpha D^\mu z^{\bar\beta} \ , \qquad (4.3)\\
\mathcal{L}_f &= \frac{1}{2}\bar\Psi_\mu\gamma^{\mu\nu\rho}D_\nu\Psi_\rho + \frac{1}{2}\bar{\overset{a}{\lambda}}\gamma^\mu D_\mu\overset{a}{\lambda} + G_{\alpha\bar\beta}\bar\psi^{\bar\beta}\gamma^\mu D_\mu L\psi^\alpha \ , \\
\mathcal{L}_{\rm int} &= \frac{1}{\sqrt{2}}G_{\alpha\bar\beta}[D_\mu z^{\bar\beta}\bar\Psi_\nu\gamma^\mu\gamma^\nu L\psi^\alpha + D_\mu z^\alpha\bar\psi^{\bar\beta}R\gamma^\nu\gamma^\mu\Psi_\nu] \\
&\quad + \frac{1}{2}\overset{a}{D}\,\bar\Psi_\mu\gamma^\mu\gamma_5\overset{a}{\lambda} + \overset{a}{F}_{\rho\sigma}\bar\Psi_\mu\gamma^{\rho\sigma}\gamma^\mu\overset{a}{\lambda} + \sqrt{2}G_{\alpha\bar\beta}[\overset{a}{X}{}^{\bar\beta}\bar{\overset{a}{\lambda}}L\psi^\alpha + \overset{a}{X}{}^\alpha\bar\psi^{\bar\beta}R\overset{a}{\lambda}] \ .
\end{aligned}$$

We omit the complicated set of four-fermion terms, see [15], but our argument includes their effects, see also the footnotes 3 and 5. We assume there is no superpotential and minimal (i.e. field independent) gauge kinetic functions to simplify the discussion. The gravitino covariant derivative is defined as

$$D_\mu\Psi_\nu \ = \ \Big(\nabla_\mu + \frac{1}{2}iB_\mu\gamma_5\Big)\Psi_\nu \ = \ \Big(\partial_\mu + \frac{1}{4}\omega_{\mu ij}\gamma^{ij} + \frac{1}{2}iB_\mu\gamma_5\Big)\Psi_\nu \ , \tag{4.4}$$

in which ∇_μ includes the spin connection. Covariant derivatives of the matter fields were given previously in (2.12, 3.5, 3.20). One must replace $\partial_\mu \longrightarrow \nabla_\mu$ in (3.20) and (3.5). Note that the composite Kähler connection (2.8) couples to all fermions.

The model has a global $U(1)_R$ axial symmetry with transformations

$$\delta L\psi^\alpha = i\alpha L\psi^\alpha \ , \quad \delta\overset{a}{\lambda} = i\alpha\gamma_5\overset{a}{\lambda} \ , \quad \delta\Psi_\mu = i\alpha\gamma_5\Psi_\mu \tag{4.5}$$

and Noether current

$$N^\mu = -\frac{i}{2}\Big[2G_{\alpha\bar\beta}\bar\psi^{\bar\beta}\gamma^\mu L\psi^\alpha + \bar{\overset{a}{\lambda}}\gamma^\mu\gamma_5\overset{a}{\lambda} + \bar\Psi_\rho\gamma^{\rho\mu\nu}\gamma_5\Psi_\nu\Big] \ . \tag{4.6}$$

It is an R-symmetry since z^α is neutral while $L\psi^\alpha$, $L\lambda^a$, and $L\Psi_\mu$ have charges $-1, +1, +1$, respectively. The $U(1)_R$ symmetry is effectively gauged by B_μ. There

is also a gauge symmetry with parameters $\theta^a(x)$ with $\delta A^a_\mu = D_\mu\theta^a$. For the gauge variation of other fields we use the notation $\delta = \theta^a\delta^a$. We then have

$$
\begin{aligned}
\overset{a}{\delta}\, z^\alpha &= \overset{a}{X}{}^\alpha\ , \quad \overset{a}{\delta}\, z^{\bar\beta} = \overset{a}{X}{}^{\bar\beta}\ , \\
\overset{a}{\delta}\, L\psi^\alpha &= \overset{a}{X}{}^\alpha{}_{,\beta}\, L\psi^\beta - \frac{i}{2}\mathrm{Im}(\overset{a}{F})L\psi^\alpha\ , \quad \overset{a}{\delta}\,\bar\psi^{\bar\beta}R = \overset{a}{X}{}^{\bar\beta}{}_{,\bar\gamma}\,\bar\psi^{\bar\gamma}R + \frac{i}{2}\mathrm{Im}(\overset{a}{F})\bar\psi^{\bar\beta}R\ , \\
\overset{b}{\delta}\ \overset{a}{\lambda} &= -f^{abc}\lambda^c - \frac{i}{2}\mathrm{Im}(\overset{b}{F})\gamma_5\lambda^a\ , \quad \overset{a}{\delta}\,\Psi_\mu = -\frac{i}{2}\mathrm{Im}(\overset{a}{F})\gamma_5\Psi_\mu\ . \qquad (4.7)
\end{aligned}
$$

Holomorphic Killing vectors $X^{a\alpha}(z)$, $X^{a\bar\beta}(\bar z)$ and the holomorphic function $F^a(z)$ induced by a gauge transformation of the Kähler potential were discussed in section 2. The gauge invariance of the theory is expressed by the identity

$$
\begin{aligned}
\delta\mathcal{L}_{\mathrm{SG}} = 0 = \overset{a}{\theta}(x)\Big[&- D_\nu\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta A^a_\nu} + \overset{a}{X}{}^\alpha\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta z^\alpha} + \overset{a}{X}{}^{\bar\beta}\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta z^{\bar\beta}} \qquad (4.8)\\
&+ \overset{a}{\delta}\,\bar\psi^{\bar\beta}R\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta\bar\psi^{\bar\beta}} + \frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta\psi^\alpha}\overset{a}{\delta}\,L\psi^\alpha + \overset{a}{\delta}\ \overset{b}{\lambda}\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta\bar\lambda^b} + \overset{a}{\delta}\,\bar\Psi_\rho\frac{\delta\mathcal{L}_{\mathrm{SG}}}{\delta\bar\Psi_\rho}\Big]
\end{aligned}
$$

which is the same as (3.3) applied to the general supergravity Lagrangian. The gauge field equation of the model reads

$$
\begin{aligned}
D_\mu\overset{a}{F}{}^{\mu\nu} &= \overset{a}{J}{}^\nu \equiv -\frac{\delta(\mathcal{L}+\frac{1}{4}F^b_{\mu\nu}F^{b\mu\nu})}{\delta A^a_\nu} \\
&= \overset{a}{J}{}^\nu_b + \overset{a}{J}{}^\nu_f + \overset{a}{j}{}^\nu + \frac{1}{2}\overset{a}{D}N^\nu + \overset{a}{J}{}^\nu_{\mathrm{int}}\ . \qquad (4.9)
\end{aligned}
$$

with J^a_b and J^a_f defined in (3.23), $j^{a\nu} = \frac{1}{2}f^{abc}\bar\lambda^b\gamma^\nu\lambda^c$ and

$$
\begin{aligned}
\overset{a}{J}{}^\nu_{\mathrm{int}} &= -\frac{\delta\mathcal{L}_{\mathrm{int}}}{\delta A^a_\nu} \\
&= \frac{1}{\sqrt{2}}G_{\alpha\bar\beta}\Big[\overset{a}{X}{}^{\bar\beta}\bar\Psi_\rho\gamma^\nu\gamma^\rho L\psi^\alpha + \overset{a}{X}{}^\alpha\bar\psi^{\bar\beta}R\gamma^\rho\gamma^\nu\Psi_\rho\Big] + 2D_\mu(\bar\Psi_\rho\gamma^{\mu\nu}\gamma^\rho\overset{a}{\lambda})\ . \qquad (4.10)
\end{aligned}
$$

To derive the consistency condition we now follow the same strategy as above. Assuming that the gauge variation of the action from varying bosons vanishes by the scalar equations of motion, the consistency condition arises from the fermion variations. The supergravity generalization of the expressions obtained earlier for only gauginos in (3.11) and for only chiral fermions in (3.27) turns out to be

$$
0 = \langle\nabla_\nu\overset{a}{J}{}^\nu\rangle = i\overset{a}{Y}_{\alpha\bar\beta}\langle\nabla_\nu(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha)\rangle + \langle\nabla_\mu\overset{a}{j}{}^\mu\rangle + \frac{1}{2}\mathrm{Im}(\overset{a}{F})\langle\nabla_\nu N^\nu\rangle\ , \qquad (4.11)
$$

in which the ∇_ν derivative carries appropriate space-time, target space and gauge connections, and $\langle...\rangle$ again indicates just the anomalous divergences of the currents.

Comparing with (4.9), we have dropped the divergence of $J^{a\nu}_{\rm int}$. As we argue in the appendix, this does not affect the anomaly. The condition (4.11) is the central result of our analysis.

The proper Kähler anomaly, proportional to $\mathrm{Im}(F^a)$, is the third term of (4.11). The anomaly has contributions from gauginos as in (3.16), from chiral fermions, as can be inferred from (3.40), and from the gravitino. The gravitino gauge anomaly is 3 times that of a gaugino, but coupled only through the Kähler connection in (4.4). We obtain

$$\langle \nabla_\nu N^\nu \rangle_{\rm gauge} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\Big[C_2(G)\overset{a}{F}_{\mu\nu}\overset{a}{F}_{\rho\sigma} + \frac{n_\lambda+3}{4}B_{\mu\nu}B_{\mu\nu} + C_{\mu\nu\rho\sigma}\Big]\ . \tag{4.12}$$

To write the contribution of the chiral fermions we first define

$$\Sigma^\alpha_{\mu\nu\beta} \;=\; R_{\gamma\bar\delta}{}^\alpha{}_\beta(D_\mu z^\gamma D_\nu z^{\bar\delta} - D_\nu z^\gamma D_\mu z^{\bar\delta})\ , \tag{4.13}$$

which is essentially the target space curvature tensor pulled back to spacetime. Using this we form

$$\begin{aligned} C_{\mu\nu\rho\sigma} \;=\;& \Sigma^\alpha_{\mu\nu\beta}\Sigma^\beta_{\rho\sigma\alpha} + \overset{a}{F}_{\mu\nu}\overset{b}{F}_{\rho\sigma}\overset{a}{X}{}^\alpha{}_{;\beta}\overset{b}{X}{}^\beta{}_{;\alpha} - \frac{1}{4}n_\psi B_{\mu\nu}B_{\rho\sigma} \\ & -i\overset{a}{F}_{\mu\nu}B_{\rho\sigma}\overset{a}{X}{}^\alpha{}_{;\alpha} - 2\overset{a}{F}_{\mu\nu}\Sigma^\beta_{\rho\sigma\alpha}\overset{a}{X}{}^\alpha{}_{;\beta} + i\Sigma^\alpha_{\mu\nu\alpha}B_{\rho\sigma}\ . \end{aligned} \tag{4.14}$$

The anomaly of the gaugino current $j^{a\mu}$ in the second term of (4.11) is identical to the truncated model of section 3.3 given in (3.17). The contribution of the chiral fermions to the first term in (4.11) is

$$\begin{aligned} \overset{a}{Y}_{\alpha\bar\beta}\langle \nabla_\nu(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha)\rangle_{\rm gauge} \;=\;& \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}G^{\beta\bar\delta}\overset{a}{Y}_{\alpha\bar\delta}\Big[\Sigma^\alpha_{\mu\nu\gamma}\Sigma^\gamma_{\rho\sigma\beta} + \overset{b}{F}_{\mu\nu}\overset{c}{F}_{\rho\sigma}\overset{b}{X}{}^\alpha{}_{;\gamma}\overset{c}{X}{}^\gamma{}_{;\beta} \\ -\frac{1}{4}B_{\mu\nu}B_{\rho\sigma}\delta^\alpha_\beta - i\overset{b}{F}_{\mu\nu}B_{\rho\sigma}\overset{b}{X}{}^\alpha{}_{;\beta} &+ i\Sigma^\alpha_{\mu\nu\beta}B_{\rho\sigma} - \overset{b}{F}_{\mu\nu}\Big(\Sigma^\alpha_{\rho\sigma\gamma}\overset{b}{X}{}^\gamma{}_{;\beta} + \Sigma^\gamma_{\rho\sigma\beta}\overset{b}{X}{}^\alpha{}_{;\gamma}\Big)\Big]\ . \end{aligned} \tag{4.15}$$

In the case of a flat target space $\mathcal{T} = \mathbb{C}^{n_\psi}$ and a linear realization of gauge symmetry, the Killing vector derivative reduces to constants, $X^{a\alpha}{}_{;\beta} = X^{a\alpha}{}_{,\beta} \to T^{a\alpha}{}_\beta$, a matrix generator of the gauge group G. In this case the second term of (4.15) reduces to the conventional cubic gauge anomaly of the chiral fermions.

The gravitational anomaly is more conventional. See Chapter 10 of [5], for example, for spin $\frac{1}{2}$ fields. The contribution of the gravitino to the anomaly of the Noether current is -21 times that of a gaugino. The gaugino current $j^{a\mu}$ itself has no gravitational anomaly. The complete result is given by

$$\begin{aligned} \langle \nabla_\nu N^\nu \rangle_{\rm grav} \;=\;& -\frac{1}{768\pi^2}(n_\lambda - 21 - n_\psi)\,\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\xi\tau}R_{\rho\sigma}{}^{\xi\tau}\ , \\ \overset{a}{Y}_{\alpha\bar\beta}\langle \nabla_\nu(\bar\psi^{\bar\beta}\gamma^\nu L\psi^\alpha)\rangle_{\rm grav} \;=\;& -\frac{i}{768\pi^2}\overset{a}{Y}_{\alpha\bar\beta}G^{\alpha\bar\beta}\,\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\xi\tau}R_{\rho\sigma}{}^{\xi\tau}\ . \end{aligned} \tag{4.16}$$

One can see that the gauge anomaly is very complicated. As a general observation, it is not possible to cancel the coefficient $n_\lambda + 3 - n_\psi$ of $B_{\mu\nu}B_{\rho\sigma}$ in (4.12) and (4.14) for the gauge anomaly and the gravitational anomaly in (4.16) at the same time by adjusting n_λ and n_ψ.

If $\mathrm{Im}(F^a) = 0$, then the $\partial_\nu N^\nu$ anomaly is absent. However, there are still several new terms involving $B_{\mu\nu}$ which can affect the consistency of the model. Anomaly cancellation will be studied in [6] with emphasis on the case of a flat target space.

5 Flux compactifications: gauged shift symmetries

In this section we apply the general formalism developed earlier to a supergravity model with a gauged shift symmetry. This form of gauge symmetry arises naturally in compactifications of ten-dimensional supergravity or string theory with background fluxes for the p-form field strengths along the internal directions [16]. Specifically, we will use a truncation of the $\mathcal{N} = 1$ flux vacuum model found in [17]. Similar structures also occur in models with gauged $\mathcal{N} = 4$ supersymmetry [18]. In the latter case one deals with a toroidal flux compactification. The gauging of shift symmetries is evident from the dimensional reduction of the type IIB 5-form, schematically[8]

$$\frac{1}{5} F^{(5)}_{MNOPR} = \partial_\mu C^{(4)}_{nopr} + 2C^{(2)}_{\mu[n} H^{(3)}_{opr]} - 2B^{(2)}_{\mu[n} F^{(3)}_{opr]} \ . \tag{5.1}$$

The kinetic term for the scalars $C^{(4)}_{nopr}$ in the four-dimensional Lagrangian then contains a coupling to the vector bosons $C^{(2)}_{\mu n}$ and $B^{(2)}_{\mu n}$, and the coupling constants are given by the values of the 3-form fluxes $F^{(3)}_{opr}$ and $H^{(3)}_{opr}$. The vectors thus gauge the shift symmetries of the scalars whenever fluxes are present.

In $\mathcal{N} = 1$ supersymmetric Calabi-Yau flux compactifications a similar gauging arises. The truncated model we will discuss includes an abelian vector multiplet with fields (A_μ, λ) and one chiral multiplet with $(L\psi, S = e^\phi + ih)$. In [17] there are several abelian gauge fields A^i_μ which arise from the reduction of the RR 4-form along 3-cycles of a Calabi-Yau manifold. They couple to scalars of chiral multiplets with charges e_i determined by flux quantum numbers. The scalar S which we retain in our truncation is the universal scalar of the string compactification. It involves the string coupling and the RR axion h. It parameterizes the well known non-compact $SU(1,1)/U(1)$ manifold.

[8]The indices are M, N, ... for ten dimensions, m, n, ... for internal six dimensions and μ, ν, ... for four dimensions. The $C^{(p)}$ are RR p-forms with field strengths $F^{(p+1)}$, $B^{(2)}$ the NSNS 2-form with field strength $H^{(3)}$.

Scalars and vectors couple through the covariant derivative $D_\mu S = \partial_\mu e^\phi + i(\partial_\mu h - eA_\mu)$. The gauge symmetry is thus the shift symmetry

$$\delta h = \theta(x) \ , \quad \delta\phi = 0 \ , \quad \delta A_\mu = \frac{1}{e}\partial_\mu \theta(x) \ . \tag{5.2}$$

The corresponding Killing vectors are imaginary constants and read

$$X^S = -X^{\bar{S}} = ie \ . \tag{5.3}$$

The Kähler potential, which is gauge invariant in this model, is

$$K = -\ln(S + \bar{S}) \ . \tag{5.4}$$

The abelian gaugino λ is always gauge invariant, and the chiral fermion ψ is gauge invariant in this model since $X^S{}_{,S} = 0$. The D field is defined by the differential equation

$$G_{S\bar{S}} X^{\bar{S}} = iD_{,S} \ , \quad G_{S\bar{S}} X^S = -iD_{,\bar{S}} \ , \tag{5.5}$$

with general solution

$$D = \frac{e}{S + \bar{S}} + \xi \ . \tag{5.6}$$

The explicit dimensional reduction of [17] leads to a potential energy from D-terms given by

$$V = \frac{1}{8} e^2 e^{-2\phi} = \frac{1}{2} D^2 \ , \tag{5.7}$$

which implies that $\xi = 0$, even though a non-vanishing value would have been compatible with $D = 4$ supergravity.

The full Lagrangian of the model is a special case of (4.2) and rather simple. It reads

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\text{int}} \ , \\
\mathcal{L}_b &= \frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - G_{S\bar{S}} D_\mu S D^\mu \bar{S} - \frac{1}{2}D^2 \ , \\
\mathcal{L}_f &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\Psi}_\mu \gamma_\nu D_\rho \Psi_\sigma + \frac{1}{2}\bar{\lambda}\gamma^\mu D_\mu \lambda + G_{S\bar{S}}\bar{\psi}\gamma^\mu D_\mu L\psi \ , \\
\mathcal{L}_{\text{int}} &= \frac{1}{\sqrt{2}} G_{S\bar{S}} [D_\mu \bar{S} \bar{\Psi}_\nu \gamma^\mu \gamma^\nu L\psi + D_\mu S \bar{\psi} R \gamma^\nu \gamma^\mu \Psi_\nu] \\
&\quad + \frac{1}{2} D \bar{\Psi}_\mu \gamma^\mu \gamma_5 \lambda + F_{\rho\sigma} \bar{\Psi}_\mu \gamma^{\rho\sigma}\gamma^\mu \lambda + \sqrt{2} G_{S\bar{S}} [X^{\bar{S}} \bar{\lambda} L\psi + X^S \bar{\psi} R\lambda] \ ,
\end{aligned} \tag{5.8}$$

where

$$
\begin{aligned}
D_\mu \Psi_\nu &= \Big(\nabla_\mu + \frac{1}{2} i B_\mu \gamma_5\Big)\Psi_\nu \ , \\
D_\mu \lambda &= \Big(\nabla_\mu + \frac{1}{2} i B_\mu \gamma_5\Big)\lambda \ , \\
D_\mu L\psi &= \Big(\nabla_\mu + \Gamma^S_{SS} D_\mu S + \frac{1}{2} i B_\mu\Big) L\psi \ .
\end{aligned} \tag{5.9}
$$

The composite Kähler connection is

$$
B_\mu = \frac{1}{2i}\Big(K_{,S}\, D_\mu S - K_{,\bar S}\, D_\mu \bar S\Big) = -\frac{1}{S+\bar S}(\partial_\mu h - e A_\mu) \ . \tag{5.10}
$$

It is gauge invariant in this model because the Kähler potential is invariant and $\mathrm{Im}(F) = \xi = 0$. One can now directly obtain the equations of motions for A_μ and h (without going through those of S and $\bar S$ first), which are

$$
\begin{aligned}
\nabla_\mu F^{\mu\nu} + 2e G_{S\bar S}(\partial^\nu h - e A^\nu) &= e J^\nu \ , \\
2\nabla_\mu (G_{S\bar S}(\partial^\mu h - e A^\mu)) &= \nabla_\mu J^\mu \ ,
\end{aligned} \tag{5.11}
$$

where

$$
J^\mu = -\frac{\delta(\mathcal{L}_f + \mathcal{L}_{\mathrm{int}})}{\delta A_\mu} \ . \tag{5.12}
$$

Applying $\frac{1}{e}\nabla_\nu$ one finds an expression which vanishes completely when the scalar equation of motion is used. Thus there is no inconsistency in this model. This agrees with the general consistency condition (4.8) because we have a gauged shift symmetry with constant Killing vectors and B_μ is gauge invariant. Therefore the fermions are invariant under the gauge transformation and all terms in (4.8) cancel when the scalar equations of motion are used. Hence no inconsistency can arise.

This favorable result depends on the particular choice of Kähler potential (5.4) which is gauge invariant under the shift symmetry. A general Kähler transformation would lead to a non-gauge invariant potential and thus a non-vanishing $\mathrm{Im}(F)$. By (4.11) this would signal that the theory is inconsistent. The resolution of this apparent problem is that one need not require Kähler invariance in this model because the target space is topologically trivial and the Kähler potential can be chosen to be gauge invariant. There is no need to consider Kähler transformations which change the preferred form (5.4).

In other situations the absence of invariance under Kähler transformations can lead to severe problems. A class of examples are toroidal orbifold compactifications

where modular $SL(2,\mathbb{Z})$ transformations of the background tori act on the moduli scalars as perturbatively exact global symmetries. They leave the Kähler potential only invariant up to Kähler transformations. The cancellation of anomalies restricts the charged matter spectrum of these models, as was for example studied in [19]. No such problem actually arises in the present case. The symmetry $S \mapsto S' = -1/S$ of the target space $SU(1,1)/U(1)$ looks potentially dangerous since it leads to a Kähler potential not invariant under a shift of S'. However, the inversion is not a symmetry of the full model because it is broken by the D-term potential (5.7) induced by the fluxes. Thus, the choice of flux breaks the global symmetry and no inconsistency arises.[9]

The problem of Kähler anomalies can reemerge if one tries to "integrate out" the scalar S replacing it with a constant background value. Then the D-term of (5.6) takes the role of a constant Fayet-Iliopoulos coupling, since S is no longer dynamical. This is the philosophy often adopted in the case of D-terms generated in the context of the Green-Schwarz mechanism. The classic example is the Fayet-Iliopoulos coupling of Dine, Seiberg, and Witten [20]. After fixing S the issue of the consistency may need to be readdressed.[10]

It is curious that if the gravitational degrees of freedom in the model of this section are dropped and the field $\mathrm{Re}(S) = e^{\phi}$ is frozen at a constant value, the model is essentially the same as that considered in section 2 of [4]. Gross and Jackiw found that the model has an axial anomaly but is consistent. It is a model of a massive vector boson in which the anomaly induces a non-renormalizable term $h\,\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$. This shows that there are models with triangle anomalies which are nevertheless consistent.

6 Discussion and Conclusions

To summarize, we have shown that the structure of anomalies in gauged non-linear σ-models coupled to supergravity is richer and more intricate than often assumed in the literature. This is mainly due to the fact that all fermions couple to the composite Kähler gauge connection B_μ. This B_μ and its field strength $B_{\mu\nu}$ depend on the scalar fields. In addition to the usual gauge, gravitational, and σ-model anomalies, $B_{\mu\nu}$ appears in the violation of gauge current conservation laws via one-loop triangle diagrams which threatens to spoil the consistency of the theory.

The Kähler transformation $K(z,\bar{z}) \to K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$ is at the root of the

[9] We would like to thank Jan Louis for this key observation.

[10] The question if S can be integrated out without breaking supersymmetry was discussed in [21].

anomaly and consistency issue. However, invariance under Kähler transformations is not necessarily required in a field theory model. We now clarify the conditions under which a significant consistency problem occurs.

1. Suppose that the σ-model target space $\mathcal{T}$ is topologically trivial, and there is a Kähler potential which is invariant under the global and gauge symmetries of the theory. Then there is a priori no need to go beyond this "preferred" $K(z,\bar{z})$, and the theory is consistent. This was the case in the flux model of section 5 where only shift symmetries were gauged.

2. If $\mathcal{T}$ is topologically non-trivial, then several coordinate charts $\mathcal{O}_A$ are required to cover $\mathcal{T}$. The Kähler potential need not be a globally defined scalar on $\mathcal{T}$; rather there is a K_A on each chart such that in overlap regions $\mathcal{O}_A \cap \mathcal{O}_B$, the potentials are related by $K_A - K_B = F_{AB}$, where F_{AB} is holomorphic. The spaces $\mathbb{CP}^n$ are examples. In this case invariance under Kähler transformation does impose consistency conditions on the supergravity model. Witten and Bagger [22] discussed important constraints even in the absence of gauging of the isometries. In addition the Kähler anomalies associated with the gauging which we have emphasized are also significant.

3. The two main situations analyzed in our paper are when $K(z,\bar{z})$ is not gauge invariant, but changes by a Kähler transformation as in (2.6), and the case of a Fayet-Iliopoulos coupling. In both cases the consistency condition (4.11) of the theory contains the extra term $\mathrm{Im}(F^a)\nabla_\nu N^\nu$ from (4.12) where N^ν is the Noether current of the global axial symmetry. This is a challenge to the consistency of all gauged supergravity theories where no gauge invariant Kähler potential exists, such as $\mathbb{CP}^n$, and to many phenomenological models that make use of Fayet-Iliopoulos couplings. Even when $\mathrm{Im}(F^a) = 0$ the consistency condition contains several new terms due to $B_{\mu\nu}$ in (4.15) which must eventually be canceled.

Conventional anomalies involve $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau}$. These structures appear in gauge current anomalies and lead to inconsistency unless cancelled. It is common practice to attempt to cancel them by adding new fermions to the model. We have found new anomaly structures which involve scalar fields, and these can require new independent cancellation conditions.

It is well known that to cancel one-loop anomalies it is possible to incorporate additional terms into a supergravity Lagrangian. This has been widely investigated in string compactifications where anomaly cancellation in the effective four-dimensional

theory is very important. See e.g. [23, 24, 25, 14, 26, 27, 28]. Let us briefly recall some aspects of the known mechanisms which have been studied. A supergravity theory may contain non-minimal field-dependent gauge kinetic functions $f_{ab}(z)$, which lead to the term

$$\mathrm{Im}(f_{ab}(z))\epsilon^{\mu\nu\rho\sigma} \overset{a}{F}_{\mu\nu} \overset{b}{F}_{\rho\sigma} \tag{6.1}$$

in the action. If $f_{ab}(z)$ is not invariant under gauge transformations, its variation contributes to the current conservation in the same way as an anomalous triangle diagram. It can cancel terms in the gauge current anomalies in (4.12) and (4.15) which are quadratic in $F^a_{\mu\nu}$. This is essentially a realization of the four-dimensional Green-Schwarz mechanism.

In [6] the anomaly cancellation conditions of supergravity models with gauge and Kähler anomalies, and with Green-Schwarz mechanism will be analyzed in the limit of a flat σ-model target space. Essential steps to obtain the physically relevant anomalies involve conversion of covariant anomalies into consistent anomalies and including all finite local counter terms in the Lagrangian. The outcome shows the necessity of a Green-Schwarz mechanism whenever the Kähler potential is not gauge invariant or Fayet-Iliopoulos couplings are present.

Other possible counter terms based on superspace integrals were proposed in [24, 25] to cancel anomalies. These terms are non-local and of the form

$$\epsilon^{\mu\nu\rho\sigma} \overset{a}{F}_{\mu\nu} \overset{a}{F}_{\rho\sigma} \frac{1}{\Box} \nabla^\rho B_\rho \ , \quad \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma} \frac{1}{\Box} \nabla^\rho B_\rho \ , \quad \ldots \tag{6.2}$$

Their gauge variation is a local expression again of the same form as induced by triangle anomalies with the respective gauge fields at the vertices. Although these terms have been studied in several string compactifications, we are not aware of any example where all anomaly structures found in our work were demonstrated to cancel.

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A Some arguments on one-loop anomalies

In the complete supergravity Lagrangian certain interaction terms were neglected in the derivation of anomalies. Here we show that there are no anomalous effect resulting from the current $J^{a\nu}_{\rm int}$ of (4.10).

We begin by discussing the gauge-fixing of the gravitino action assuming a flat space-time background for simplicity. Let's add a gauge-fixing term to the gravitino kinetic Lagrangian which includes the B_μ connection, obtaining

$$\mathcal{L} = \frac{1}{2}(\bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho + \zeta \bar{\Psi}_\mu \gamma^\mu \gamma^\nu D_\nu \gamma^\rho \Psi_\rho) \ . \tag{A.1}$$

In the vNV gauge [29], in which $\zeta = -\frac{1}{2}$, this becomes

$$\mathcal{L} = -\frac{1}{4} \bar{\Psi}_\mu \gamma^\rho \gamma^\nu D_\nu \gamma^\mu \Psi_\rho \ . \tag{A.2}$$

The linear field redefinition [30]

$$\Psi_\mu \mapsto U_\mu^\nu \Psi_\nu \ , \quad U_\mu^\nu = \delta_\mu^\nu - \frac{1}{2}\gamma_\mu \gamma^\nu \ , \quad U_\mu^\rho U_\rho^\nu = \delta_\mu^\nu \tag{A.3}$$

takes us to the ultra-simple AGW gauge [31] Lagrangian

$$\mathcal{L} = \frac{1}{2} \bar{\Psi}^\mu \gamma^\nu D_\nu \Psi_\mu \ . \tag{A.4}$$

Using standard γ-matrix identities the current can be written in terms of the new variable Ψ_μ as

$$\overset{a}{J}{}^\nu_{\rm int} = \sqrt{2} G_{\alpha\bar{\beta}} \Big(\overset{a}{X}{}^{\bar{\beta}} \bar{\Psi}^\nu L \psi^\alpha + \overset{a}{X}{}^\alpha \bar{\psi}^{\bar{\beta}} R \Psi^\nu \Big) + 2 D_\mu \Big(\bar{\Psi}_\rho \gamma^{\mu\nu} \gamma^\rho \overset{a}{\lambda} \Big) \ . \tag{A.5}$$

Things simplify and there is little loss of generality if we study the situation in the truncated Grimm-Louis model of section 5 in which $L\psi^\alpha$ can be replaced by the single Majorana spinor ψ and the total fermion current of the model is

$$\begin{aligned} J^\nu &= -\frac{i}{4} \frac{e}{S+\bar{S}} \Big(\bar{\lambda} \gamma^\nu \gamma_5 \lambda + \bar{\Psi}^\mu \gamma^\nu \gamma_5 \Psi_\mu - \frac{5}{(S+\bar{S})^2} \bar{\psi} \gamma^\nu \gamma_5 \psi \Big) + J^\nu_{\rm int} \ , \\ J^\nu_{\rm int} &= \sqrt{2} \frac{ie}{(S+\bar{S})^2} (\bar{\Psi}^\nu \gamma_5 \psi) + 2 \partial_\mu (\bar{\Psi}_\rho \gamma^{\mu\nu} \gamma^\rho \lambda) \ . \end{aligned} \tag{A.6}$$

We look for possibly anomalous triangle diagrams for the 3-point function of currents $\langle J^\mu(z)J^\nu(y)J^\rho(y)\rangle$ in which the current $J^{a\mu}_{\rm int}$ appears in at least one position. Among diagrams with an internal ψ line (and no gauginos), one can rapidly see that the either Wick contractions vanish or the diagrams have vanishing γ-matrix trace.

There are two non-vanishing one-loop diagrams involving two insertions of the gaugino part of $J^{a\mu}_{\rm int}$ and an insertion of the gaugino and (gauge-fixed) gravitino axial currents. These diagrams are complicated and a regulated calculation appears to be difficult. Therefore we adopted another strategy, in which we generalize the Fujikawa analysis to include the mixing of a gravitino and (abelian) gaugino in the Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\bar{\Psi}^\mu\gamma^\nu D_\nu\Psi_\mu + \frac{1}{2}\bar{\lambda}\gamma^\nu D_\nu\lambda + \bar{\Psi}_\mu\gamma^{\rho\sigma}F_{\rho\sigma}\gamma^\mu\lambda\ , \tag{A.7}$$

with $D_\nu = \partial_\nu + \frac{1}{2}iB_\nu\gamma_5$. We then defined the operators $\mathcal{D}_L = \mathcal{D}L$ and $\mathcal{D}_R = \mathcal{D}R$ in which $\mathcal{D}$ is the matrix operator

$$\mathcal{D} = \begin{pmatrix} \not{D}\delta^\nu_\mu & F_{\alpha\beta}\gamma^{\alpha\beta}\gamma_\mu \\ \gamma^\nu F_{\alpha\beta}\gamma^{\alpha\beta} & \not{D} \end{pmatrix}\ . \tag{A.8}$$

It acts on $(\Psi_\nu,\lambda)^T$ to the right and on $(\bar{\Psi}^\mu,\bar{\lambda})$ to the left. Mode expansions lead to a Jacobian in the Fujikawa method which is a generalization of that of sections 6.4 and 6.4.1 of [5] and reads

$$\ln(J) = -2i\int d^4k\,\alpha(x)e^{-ik\cdot x}{\rm tr}\Big[f\Big(\frac{\mathcal{D}\mathcal{D}}{M^2}\Big)\gamma_5\Big]e^{ik\cdot x}\ . \tag{A.9}$$

The function $f(\mathcal{D}\mathcal{D}/M^2)$ is a smoothly decreasing function of the mode eigenvalues with cutoff scale M^2. After shifting $\not{D}$ by the plane wave momentum k_μ and expanding in powers of $1/M$ we find that potentially non-vanishing terms of order up to $1/M^4$ exist. In the standard calculation with $\not{D}$ instead of $\mathcal{D}$ one only has to evaluate the product $\not{D}\not{D}$ and extract the term $(\gamma^{\alpha\beta}F_{\alpha\beta})^2$ that has a non-vanishing trace with γ_5. Here it becomes necessary to compute $\mathcal{D}^4$ and consider many independent contributions. The calculation is too tedious to report in detail. Suffice it to say that, by careful evaluation of traces, we were able to show that all effects of the $\bar{\Psi}_\mu\gamma^{\rho\sigma}F_{\rho\sigma}\gamma^\mu\lambda$ mixing term vanish, and the only contribution to the trace comes from the conventional terms $(\not{D}\not{D})^2$. The gauge anomaly reduces (after inclusion of gravitino ghosts) to the well known anomaly of $\mathcal{N}=1$ supergravity coupled to one gauge multiplet.

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